World Spinors - Construction and Some Applications

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ABSTRACT

The existence of a topological double-covering for the $GL(n, R)$ and diffeomorphism groups is reviewed. These groups do not have finite-dimensional faithful representations. An explicit construction and the classification of all $\overline{SL}(n, R)$, $n = 3, 4$ unitary irreducible representations is presented. Infinite-component spinorial and tensorial $\overline{SL}(4, R)$ fields, "manifields", are introduced. Particle content of the ladder manifields, as given by the $\overline{SL}(3, R)$ "little" group is determined. The manifields are lifted to the corresponding world spinorial and tensorial manifields by making use of generalized infinite-component frame fields. World manifields transform w.r.t. corresponding $\overline{Diff}(4, R)$ representations, that are constructed explicitly.

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1. INTRODUCTION

Larry Biedenharn's contributions to physics span several of its subdisciplines, such as Atomic, Nuclear, or Particle Physics. The common denominator is his masterly handling of Group Theory, certainly a very powerful tool in these fields. One of the most beautiful examples of Larry's virtuoso performance is his solution [1] of the Racah problem: How does one characterize - with no degeneracies - the states in the unitary irreducible representations of $SU(3)$, when applying (e.g. as in harmonic oscillator models) the $SU(3) \rightarrow SO(3)$ reduction sequence, i.e. with the $SO(3)$ 3-dimensional vector spanning the same carrier space as the 3-dimensional defining representation of $SU(3)$. The problem caught the interest and imagination of the algebraic experts (including Racah himself), who worked on it, with the late Y. Lehrer-Ilamed for years. L.B.'s solution is "final" and also shows that there are no rational operator functions capable of fulfilling the task, while presenting the irrational functions which do.

The authors of this article owe their mutual links, which produced their intensive twenty years' personal collaboration, to the fact that their careers intersected with Larry Biedenharn's, in the group theory context. The first author (YN) while at Caltech in 1963-65, happened to produce, in collaboration with M. Gell-Mann and with the late Yossef Dothan, a suggestion for a group-theoretical characterization of the hadron Regge sequences, till then charted phenomenologically, after the great resonance "explosion" in 1960-61 [2]. The model also supplied an algebraic structural derivation, involving *gravitational quadrupoles*. This appeared rather surprising at the time, but has been explained by the present authors in recent years [3]. The algebraic Regge model [2], based on assignments to *ladder*-type infinite representations of the noncompact group $SL(3,R)$ (whose construction was also first given in [2]), with $\Delta J = 2$ and for lowest spins $J_0 = 0, 1, 2$, also appeared to be extendable to nuclear physics. This is a part of Physics in which "quadrupolar" algebras (based on the harmonic oscillator $SU(3)$ degeneracy group) had been introduced by Elliott in 1958 [4].

YN first met Larry Biedenharn at the Coral Gables Conferences, and discussed these $SL(3,R)$ results and their possible relevance to Nuclei. Larry was interested and several

years later (1970-73) indeed successfully applied the $SL(3,R)$ algebra to nuclei [5]. Indeed, one now has a good understanding [6] of the intertwining of the three different algebras $(SL(3,R), SU(3), SO(3) \times T_(3))$ which can be generated by the commutators between angular momentum and quadrupole operators.

And yet $SL(3,R)$ went "deeper". The question of *the existence of a double covering* had already arisen in 1965, when the authors of [2] looked for an $SL(3,R)$ assignment, to fit the fermionic Regge trajectories. This had, however, remained unanswered. In 1969, when YN was next at Caltech for a term, he initiated an algebraic study of the case, together with Dr D.W. Joseph, of the University of Nebraska, with whom he had collaborated in 1964 in a Kaluza-Klein approach to (flavor) $SU(3)$. The answer to the question of the existence of a double-covering was indeed positive, there is such a $\overline{SL}(3,R)$ group with only infinite unitary representations and one should thus have been able to utilize these unitary irreducible infinite representations for fermionic sequences. However, an unexpected difficulty suddenly emerged in this program, in the form of a *singularity* occurring in the "ladder"-like representation whose lowest state is $J_0 = 3/2$ (needed for the "most important" hadron resonance, Fermi's $(I = 3/2, J = 3/2)$). Note that there was no difficulty with $J_0 = 1/2$. David Joseph prepared a preprint for publication [7], but the enthusiasm for publication had waned for YN, as the answer appeared to fail for the most important physical case, the $I = 3/2, J = 3/2$. Joseph sent out his preprint, which was never published, as a result of a combination of referee difficulties and loss of enthusiasm. However, the preprint did trigger a renewal of interest in $SL(3,R)$ among the group theory fans, including in Larry's group at Duke University [5]. The difficulty with $J_0 = 3/2$ was first glossed over, but then resurfaced, with a contribution [8] from another group-theory virtuoso, the late V. Ogievetsky, who died in the same year 1995 as Larry Biedenharn, in a sports accident.

Meanwhile, the second author (Dj. Š.) had arrived in 1972 at Duke University, becoming engaged in a doctoral dissertation program. With the interest in $SL(3,R)$ as displayed in both particle and nuclear physics, it seemed worth investing a real effort in charting the entire system of representations, including those of the double-covering. D.S.'s results, published in 1975 [9], were extensive and "final", as emphasized several years later

in a more mathematically oriented study [10]. Thus, when in 1977, YN demonstrated [11, 12, 13] the relevance of these results to an issue in Gravity, namely the erroneous ruling-out of curved space spinors (*world spinors*), it was natural that the two authors should converge in their interests - and the present collaboration was born.

We now present the problem from that gravitational angle.

In the standard approach to General Relativity one starts with the group of "general coordinate transformations" (GCT), i.e. the group of diffeomorphisms Diff(R^4). The theory is set upon the principle of general covariance. A unified description of both tensors and spinors would require the existence of respectively tensorial and (double valued) spinorial representations of the GCT group. In other words one is interested in the corresponding single-valued representations of the double covering $\overline{GCT}$ of the GCT group, since the topology of GCT is given by the topology of its linear compact subgroup. It is well known that the finite-dimensional representations of $\overline{GCT}$ are characterized by the corresponding ones of the $\overline{GL}(4,R) \supset \overline{SL}(4,R)$ group, and $\overline{SL}(4,R)$ does not have *finite* spinorial representations. However there are infinite-dimensional spinors of $\overline{SL}(4,R)$ which are the true "world" (holonomic) spinors [14]. There are two ways to introduce finite spinors: i) One can make use of the nonlinear representations of the $\overline{GCT}$ group, which are linear when restricted to the Poincaré subgroup [15]. ii) One can introduce a bundle of cotangent frames, i.e. a set of 1-forms e^a (tetrads; $a = 0, ..., 3$ the anholonomic indices) and define in this space an action of a physically distinct local Lorentz group. Owing to this Lorentz group one can introduce finite spinors, which behave as scalars w.r.t. $\overline{GCT}$. The bundle of cotangent frames represents an additional geometrical construction corresponding to the physical constraints of a local gauge group of the Yang-Mills type, in which the gauge group is the isotropy group of the space-time base manifold. One is now naturally led to enlarge the local Lorentz group to the whole linear group $\overline{GL}(4,R)$, and together with translations one obtains the affine group $\overline{GA}(4,R)$. The affine group translates and deforms the tetrads of the locally Minkowskian space-time [16], and provides one with either infinite-dimensional linear or finite-dimensional nonlinear spinorial representations [17].

The existence and structure of spinors in a generic curved space have been the subject of more confusion than most issues in mathematical physics. True, to the algebraic

topologist the problem appears to have been answered long ago, with the realization that *the topology of a noncompact Lie group follows that of its maximal compact subgroup.* This perhaps is the reason for the low priority given by mathematicians, in the case of the linear groups, to the study of the representations of their double-covering, for instance [10].

The issue is an important one for the physicist, however, and we shall make one more effort to clarify it. The physics literature contains two common errors. For fifty years, it was wrongly believed that the double-covering of $GL(n, R)$, which we shall denote $\overline{GL}(n, R)$ does not exist. Almost every textbook in general relativity theory, upon reaching the subject of spinors, contains a sentence such as "... there are no representations of $GL(4, R)$, or even 'representations up to a sign', which behave like spinors under the Lorentz subgroup". Though the correct answer has been known since 1977 [11, 12, 13], the same type of statement continues to appear in more recent texts. The present authors were much encouraged in their dealing with the issue of spinors in a curved space by the convergence of their interests in this matter with the investigation of Metric-Affine manifolds initiated by F.W. Hehl and his Cologne group [16]. The contents of a recent review [18] testify to the richness of the subject.

An additional reason for the overall confusion concerns the unitarity of the relevant spinor representations. In dealing with noncompact groups, it is customary to select infinite-dimensional unitary representations, where the *particle-states* are concerned. For both tensor or spinor *fields*, however, finite and nonunitary representations are used (of $GL(4, R)$ and $SL(2, C)$ respectively). We showed that the correct answer for spinorial $\overline{GL}(4, R)$ fields consists in using the infinite unitary representations in a physical base in which they become nonunitary [19].

In recent years, the unitary infinite-dimensional representations of the double-coverings $\overline{GL}(n, R)$ and $\overline{SL}(n, R)$ have been classified and constructed for $n = 3$ [9], $n = 4$ [20], while the case $n = 2$ has been known for many years [21]. Field equations have been constructed for such infinite-component fields, "manifields", within Riemannian gravitational theory and for Einstein-Cartan gravity [22], including the case of "world spinors" [14], and for affine [17, 23, 24] gravity. $\overline{SL}(4, R)$ manifields have also been used in classifying the hadron spectrum [25, 26].

2. EXISTENCE OF THE DOUBLE-COVERING $\overline{GL}(n, R)$

The basic results can be found in Ref. [27]. Let $g_0 = k_0 + a_0 + n_0$ be an Iwasawa decomposition of a semisimple Lie algebra g_0 over R. Let G be any connected Lie group with Lie algebra g_0, and let K, A, N be the analytic subgroups of G with Lie algebras k_0,a_0 and n_0 respectively. The mapping $(k, a, n) \to kan \quad (k \in K, a \in A, n \in N)$ is an analytic diffeomorphism of the product manifold $K \times A \times N$ onto G. The groups A and N are simply connected. Any semisimple Lie group can be decomposed into the product of the maximal compact subgroup K, an Abelian group A and a nilpotent group N. As a result of, only K is not guaranteed to be simply-connected. There exists a universal covering group $\overline{K}_u$ of K, and thus also a universal covering of G:

$$\overline{G}_u \simeq \overline{K}_u \times A \times N.$$

For the group of diffeomorphisms, let $Diff(n, R)$ be the group of all homeomorphisms f or R^n such that f and f^{-1} are of class C^1. In the neighborhood of the identity

$$V_{r,\varepsilon} = \left\{ g \in Diff(n, R) \mid [g(x) - x] < \varepsilon, [\frac{\partial g_i}{\partial x_k}(x) - \delta^i_k] < \varepsilon, \mid x \mid < r \quad i, k = 1, ..., n \right\}$$

Stewart [28] proved the decomposition

$$Diff(n, R) = GL(n, R) \times H \times R_n$$

where the subgroup H is contractible to a point. As $O(n)$ is the compact subgroup of $GL(n, R)$, one finds that $O(n)$ is a deformation retract of $Diff(n, R)$. Thus, there exists a universal covering of the Diffeomorphism group

$$\overline{Diff}(n, R)_u \simeq \overline{GL}(n, R)_u \times H \times R_n.$$

Summing up, we note that both $SL(n, R)$ and on the other hand $GL(n, R)$ and $Diff(n, R)$ will all have double coverings, defined by $\overline{SO}(n) \simeq Spin(n)$ and $\overline{O}(n) \simeq Pin(n)$ respectively, the double-coverings of the $SO(n)$ and $O(n)$ maximal compact subgroups.

3. $\overline{SL}(3, R)$ AND $\overline{SL}(4, R)$ UNIRREPS

$SL(n,R)$ is the group of linear unimodular transformations in an n-dimensional real vector space. The group is a simple and noncompact Lie group. The space of the group parameters is not simply connected. The maximal compact subgroup of $SL(n,R)$ is $SO(n)$. The double covering (the universal covering for $n>2$) group of $SL(n,R)$ we denote by $\overline{SL}(n,R)$. Its maximal compact subgroup is $\overline{SO}(n) \simeq Spin(n)$, the covering group of $SO(n)$.

$$\overline{SL}(n,R)/Z_2 \simeq SL(n,R), \quad \overline{SO}(n)/Z_2 \simeq SO(n).$$

In order to present the explicit forms of the $\overline{SL}(n,R)$ generators, $n=3,4$, we first separate them according to compactness and it is most convenient to take them in the spherical basis. We list a minimal set of commutation relations. The remaining ones can be obtained by means of the Jacobi identity.

The $\overline{SL}(3,R)$ generators are $J_0,\ J_\pm,\ T_M,\ M=0,\pm1,\pm2$. J_0 and $J_\pm$ generate the $SU(2)$ subgroup, while T_M forms an $SU(2)$ second rank irreducible tensor operator. The commutation relations are:

$$\begin{aligned}
&[J_0,J_\pm]=\pm J_\pm, \quad [J_+,J_-]=2J_0, \quad [J_0,T_M]=MT_M,\\
&[J_\pm,T_M]=\sqrt{6-M(M\pm1)}T_{M\pm1} \quad [T_{+2},T_{-2}]=-4J_0.
\end{aligned}$$

The $\overline{SL}(4,R)$ generators are $J_0^{(i)},\ J_\pm^{(i)},\ Z_{pq},\ i=1,\ 2;\ p,\ q=0,\ \pm1$. $J_0^{(i)}$ and $J_\pm^{(i)}$ generate the $SU(2)\otimes SU(2)$ subgroup, while Z_{pq} forms, w.r.t. $SU(2)\otimes SU(2)$, a $(1,1)$-irreducible tensor operator. The commutation relations are:

$$\begin{aligned}
&[J_0^{(i)},J_\pm^{(j)}]=\pm\delta_{ij}J_\pm^{(i)}, \quad [J_+^{(i)},J_-^{(j)}]=2\delta_{ij}J_0^{(i)}, \quad [J_0^{(i)},J_0^{(j)}]=0\\
&[J_0^{(1)},Z_{pq}]=pZ_{pq}, \quad [J_0^{(2)},Z_{pq}]=qZ_{pq}\\
&[J_\pm^{(1)},Z_{pq}]=\sqrt{2-p(p\pm1)}Z_{p\pm1,q}, \quad [J_\pm^{(2)},Z_{pq}]=\sqrt{2-q(q\pm1)}Z_{p,q\pm1},\\
&[Z_{+1,+1},Z_{-1,-1}]=-(J_0^{(1)}+J_0^{(2)}).
\end{aligned}$$

In order to analyze the representations, as well as to make use of them in a gauge theory, it is convenient to have the matrix elements of the group generators. Also, in this case the task of determining the scalar products of the unitary representations is considerably simplified. The most general results are obtained in the $\left|\begin{smallmatrix} & j & \\ k & & m\end{smallmatrix}\right\rangle$, $\left|\begin{smallmatrix} & j_1 & & & j_2 & \\ k_1 & & m_1 & k_2 & & m_2\end{smallmatrix}\right\rangle$ basis of the $SU(2)$, $SU(2)\otimes SU(2)$ representations respectively, $j,\ j_1,\ j_2=0,\ 1/2,\ 1\ \ldots$

The matrix elements of the compact generators are well known, and we list only the matrix elements of the noncompact generators [10, 20].

n=3:

$$\left\langle \begin{matrix} j' \\ k'\ m' \end{matrix} \middle| T_M \middle| \begin{matrix} j \\ k\ m \end{matrix} \right\rangle = (-)^{j'-m'} \begin{pmatrix} j' & 2 & j \\ -m' & M & m \end{pmatrix} \left\langle \begin{matrix} j' \\ k' \end{matrix} \middle\| T \middle\| \begin{matrix} j \\ k \end{matrix} \right\rangle, \quad M = 0, \pm 1, \pm 2,$$

where,

$$\left\langle \begin{matrix} j' \\ k' \end{matrix} \middle\| T \middle\| \begin{matrix} j \\ k \end{matrix} \right\rangle = (-)^{j'-k'} \sqrt{(2j'+1)(2j+1)} \left\{ \frac{-i}{\sqrt{6}} [2\sigma - j'(j'+1) + j(j+1)] \begin{pmatrix} j' & 2 & j \\ -k' & 0 & k \end{pmatrix} + \right.$$

$$\left. + i(\delta + k + 1) \begin{pmatrix} j' & 2 & j \\ -k' & 2 & k \end{pmatrix} + i(\delta - k + 1) \begin{pmatrix} j' & 2 & j \\ -k' & -2 & k \end{pmatrix} \right\},$$

$\sigma = a + b, \ \delta = a - b.$

n=4:

$$\left\langle \begin{matrix} j_1' & & j_2' & \\ k_1' & m_1' & k_2' & m_2' \end{matrix} \middle| Z_{pq} \middle| \begin{matrix} j_1 & & j_2 & \\ k_1 & m_1 & k_2 & m_2 \end{matrix} \right\rangle =$$

$$= (-)^{j_1'-m_1'} (-)^{j_2'-m_2'} \begin{pmatrix} j_1' & 1 & j_1 \\ -m_1' & p & m_1 \end{pmatrix} \begin{pmatrix} j_2' & 1 & j_2 \\ -m_2' & q & m_2 \end{pmatrix} \left\langle \begin{matrix} j_1'\ j_2' \\ k_1'\ k_2' \end{matrix} \middle\| Z \middle\| \begin{matrix} j_1\ j_2 \\ k_1\ k_2 \end{matrix} \right\rangle,$$

where,

$$\left\langle \begin{matrix} j_1'\ j_2' \\ k_1'\ k_2' \end{matrix} \middle\| Z \middle\| \begin{matrix} j_1\ j_2 \\ k_1\ k_2 \end{matrix} \right\rangle = (-)^{j_1'-k_1'} (-)^{j_2'-k_2'} \frac{i}{2} \sqrt{(2j_1'+1)(2j_2'+1)(2j_1+1)(2j_2+1)} \times$$

$$\times \left\{ [e + 4 - j_1'(j_1'+1) + j_1(j_1+1) - j_2'(j_2'+1) + j_2(j_2+1)] \begin{pmatrix} j_1' & 1 & j_1 \\ -k_1' & 0 & mk_1 \end{pmatrix} \begin{pmatrix} j_2' & 1 & j_2 \\ -k_2' & 0 & k_2 \end{pmatrix} \right.$$

$$- (c + k_1 - k_2) \begin{pmatrix} j_1' & 1 & j_1 \\ -k_1' & 1 & mk_1 \end{pmatrix} \begin{pmatrix} j_2' & 1 & j_2 \\ -k_2' & -1 & k_2 \end{pmatrix}$$

$$- (c - k_1 + k_2) \begin{pmatrix} j_1' & 1 & j_1 \\ -k_1' & -1 & mk_1 \end{pmatrix} \begin{pmatrix} j_2' & 1 & j_2 \\ -k_2' & 1 & k_2 \end{pmatrix}$$

$$+ (d + k_1 + k_2) \begin{pmatrix} j_1' & 1 & j_1 \\ -k_1' & 1 & mk_1 \end{pmatrix} \begin{pmatrix} j_2' & 1 & j_2 \\ -k_2' & 1 & k_2 \end{pmatrix}$$

$$\left. + (d - k_1 - k_2) \begin{pmatrix} j_1' & 1 & j_1 \\ -k_1' & -1 & mk_1 \end{pmatrix} \begin{pmatrix} j_2' & 1 & j_2 \\ -k_2' & -1 & k_2 \end{pmatrix} \right\},$$

$e = c - a - b, \ d = a - b.$

The representation labels σ, δ (for $n = 3$); and c, d, e (for $n = 4$) are arbitrary complex numbers and are determined from the representation space scalar product's unitarity and from the group generators' hermiticity requirements.

We now list all unitary irreducible representation labels and the $\overline{SO}(3, R)$ subgroup labels of the $\overline{SL}(3, R)$ group [9].

Principal series: $\sigma_1 = \delta_1 = 0, \quad \sigma_2, \delta_2 \in R$

$$(\varepsilon, \varepsilon') = (+1, +1): \ \{j\} = \{0^1, 2^2, 3^1, 4^3, 5^2, ...\}$$

$$(\varepsilon, \varepsilon') = (+1, -1), \ (-1, \pm 1): \ \{j\} = \{1^1, 2^1, 3^2, 4^2, 5^3, ...\}, \ \{\tfrac{1}{2}^1, \tfrac{3}{2}^2, \tfrac{5}{2}^3, ...\}.$$

Supplementary series: $\sigma_1 = \delta_2 = 0, \quad \sigma_2 \in R$

$$0 < \delta_1 < 1, \quad (\varepsilon, \varepsilon') = (+1, +1): \ \{j\} = \{0^1, 2^2, 3^1, 4^3, 5^2, ...\}$$

$$(\varepsilon, \varepsilon') = (+1, -1): \ \{j\} = \{1^1, 2^1, 3^2, 4^2, 5^3, ...\}$$

$$0 < \delta_1 \leq \tfrac{1}{2}, \quad \{j\} = \{\tfrac{1}{2}^1, \tfrac{3}{2}^2, \tfrac{5}{2}^3, ...\}$$

Discrete series: $\sigma_1 = \delta_2 = 0, \ \sigma_2 \in R, \ \delta_1 = 1 - \underline{j}; \ \underline{j} = \frac{3}{2}, 2, \frac{5}{2}, 3, ...$

$$\{j\} = \{\underline{j}^1, (\underline{j}+1)^1, (\underline{j}+2)^2, (\underline{j}+3)^2, (\underline{j}+4)^3, ...\}$$

Multiplicity free (ladder) series: $\sigma_1 = \delta_2 = 0, \ \delta_1 = 1,$

$$\sigma_2 \in R, \quad \{j\} = \{0, 2, 4, ...\}, \ \{j\} = \{1, 3, 5, ...\}$$

$$\sigma_2 = 0, \quad \{j\} = \{\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{9}{2}, ...\}.$$

For the general case of the $\overline{SL}(4, R)$ unirreps we present here only the labels. For the general (multiplicity non free) case, we have [20]

A) $e_1 = 0, \ e_2 \in R,$

B$_1$) $d_1 = 0, \ d_2 \in R,$

B$_2$) $d_1 = \underline{k}_1 + \underline{k}_2, \ d_2 = 0; \quad \underline{k}_1 + \underline{k}_2 = \frac{1}{2}, 1, \frac{3}{2}, ...,$

B$_3$) $0 < d_1 < 1, \ d_2 = 0; \quad k_1 + k_2 = 0, \pm 2, \pm 4, ...,$

B$_4$) $0 < d_1 < \frac{1}{2}, \ d_2 = 0; \quad k_1 + k_2 \equiv \frac{1}{2}(mod2) \quad$ or $\quad \frac{3}{2}(mod2),$

C$_1$) $c_1 = 0, \ c_2 \in R,$

C$_2$) $c_1 = \underline{k_1} - \underline{k_2}, \ c_2 = 0; \quad \underline{k_1} - \underline{k_2} = \frac{1}{2}, 1\frac{3}{2}, ...,$

C$_3$) $0 < c_1 < 1, \ c_2 = 0; \quad k_1 - k_2 = 0, \pm 2, \pm 4, ...,$

C$_4$) $0 < c_1 < \frac{1}{2}, \ c_2 = 0; \quad k_1 - k_2 = \frac{1}{2}(mod2) \quad$ or $\quad \frac{3}{2}(mod2).$

Any combination of (A) with one (B) and one (C) determines a series of $\overline{SL}(4, R)$ unirreps. For these series $j_1 \geq |k_1|, \ j_2 \geq |k_2|$. There are four series of multiplicity free $\overline{SL}(4, R)$

unirreps [19].

Principal series: $e_1 = 0,\ e_2 \in R;\quad j_1 + j_2 \equiv 0(mod2)\quad \text{or}\quad 1(mod2)$,

Supplementary series: $0 < e_1 < 1,\ e_2 = 0;\quad j_1 + j_2 \equiv 1(mod2)$,

Discrete series: $e_1 = 1 - \underline{j},\ e_2 = 0;\quad \underline{j} = \frac{1}{2}, 1, \frac{3}{2}, ..., |j_1 - j_2| \geq \underline{j}$,

Ladder series: $e_1 = 0,\ e_2 \in R;\ j_1 = j_2 = j,\ \{j\} = \{0, 1, 3, ...\}, \{j\} = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}...\}$.

4. $\overline{GA}(4, R)$ OR $\overline{SA}(4, R)$ MATTER FIELDS.

The general affine group $\overline{GA}(4, R) = T_4 \circledS \overline{GL}(4, R)$, is a semidirect product of translations and $\overline{GL}(4, R)$, the general linear group, generated by Q_{ab}. Here $\overline{GL}(4, R) = R_+ \otimes \overline{SL}(4, R) \supset R_+ \otimes \overline{SO}(1, 3)$, where R_+ is the dilation subgroup. The antisymmetric operators $Q_{[ab]} = \frac{1}{2}(Q_{ab} - Q_{ba})$ generate the Lorentz subgroup $\overline{SO}(1, 3)$, the symmetric traceless operators (shears) $Q_{(ab)} = \frac{1}{2}(Q_{ab} + Q_{ba}) - \frac{1}{4} g_{ab} Q_c{}^c$ generate the proper 4-volume-preserving deformations while the trace $Q = Q^a{}_a$ generates scale-invariance R_+. $Q_{[ab]}$ and $Q_{(ab)}$ generate together the $\overline{SL}(4, R)$ group.

The $\overline{SA}(4, R)$ unirreps [19, 20] are induced from the corresponding little group unirreps. The little group turns out to be $\overline{SA}(3, R)^{\sim} = T_3^{\sim} \circledS \overline{SL}(3, R)$, and thus we have the following nontrivial possibilities:

(i) $T_3^{\sim}$ is represented trivially, and the corresponding states are described by the $\overline{SL}(3, R)$ unirreps, which are infinite-dimensional owing to the $\overline{SL}(3, R)$ noncompactness. The corresponding $\overline{SL}(4, R)$ matter fields are therefore necessarily infinite-dimensional and when reduced with respect to the $\overline{SL}(3, R)$ subgroup should transform with respect to its unirreps.

(ii) The little group $\overline{SA}(3, R)^{\sim}$ is represented nontrivially, and we find the states which are characterized "effectively" by three real numbers in addition to the $\overline{SA}(2, R)^{\sim}$ unirreps.

(iii) For quarks or leptons, we make use of the $\overline{GA}(4, R)$ nonlinear representations which are realized through metric g_{ab}. The stability subgroup is $SL(2, C)$, and the representations are linear for the Poincaré subgroup.

Had the whole $\overline{SL}(4, R)$ been represented unitarily, the Lorentz boost generators

would have a hermitian intrinsic part; as a result, when boosting a particle, one would obtain a particle with a different spin, i.e. another particle - contrary to experience. There exists however a remarkable inner *deunitarizing* automorphism $\mathcal{A}$ [19], which leaves the $R_+ \otimes \overline{SL}(3,R)$ subgroup intact, and which maps the $Q_{(0k)}$, $Q_{[0,k]}$ generators into $iQ_{[0k]}$, $iQ_{(0k)}$ respectively ($k=1,2,3$). The deunitarizing automorphism allows us to start with the unitary representations of the $\overline{SL}(4,R)$ subgroup, and upon its application, to identify the finite (unitary) representations of the abstract $\overline{SO}(4,R)$ compact subgroup with nonunitary representations of the physical Lorentz group, while the infinite (unitary) representations of the abstract $\overline{SO}(1,3)$ group now represent (non-unitarily) the compact $\overline{SO}(4)/\overline{SO}(3)$ generators. The non-hermiticity of the intrinsic boost operators cancels their "intrinsic" physical action precisely as in finite tensors or spinors, the boosts thus acting kinetically only. In this way, we avoid a disease common to infinite-component wave equations.

Let us denote a generic $\overline{SL}(4,R)$ unirrep by $D(c,d,e;(j_1,j_2))$ where c, d, e are the representation labels, and (j_1,j_2) denote the lowest $\overline{SO}(4) = SU(2) \otimes SU(2)$ representation contained in the given $\overline{SL}(4,R)$ representation.

For the $\overline{SL}(4,R)$ tensorial field we take an infinite-component field Φ which transforms with respect to an $\mathcal{A}$-deunitarized unirrep belonging to the principal series of representations $D^{pr}_{SL(4,R)}(c_2,d_2,e_2,(00))$, c_2, d_2, $e_2 \in R$. The manifield Φ obeys a Klein-Gordon-like equation

$$\left(g^{ab}\partial_a\partial_b + M^2\right)\Phi(x) = 0.$$

For the $\overline{SL}(4,R)$ spinorial fields we take an infinite-component field Ψ which transforms with respect to an $\mathcal{A}$ deunitarized unirrep belonging to the principal series of representations: $D^{pr}_{\overline{SL}(4,R)}(c_2,d_2,e_2;(\frac{1}{2},0)) \oplus D^{pr}_{\overline{SL}(4,R)}(c_2,d_2,e_2;(0,\frac{1}{2}))$, $c_2,d_2,e_2 \in R$ while $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ denote parity-conjugated spinorial representations. The manifield Ψ satisfies a Dirac-like equation

$$\left(ig^{ab}\chi_a\partial_b - M\right)\Psi(x) = 0,$$

where χ_a is an $\overline{SL}(4,R)$ four-vector acting in the space of our spinorial manifield. We construct χ_a in the following way: first we embed $\overline{SL}(4,R)$ into $\overline{SL}(5,R)$, and then se-

lect a pair of (mutually conjugate) principal series representations which contain in the $\overline{SL}(4,R)$ reduction our spinorial representations. Let the $\overline{SL}(5,R)$ generators be $Q_{\hat{a}\hat{b}}$, $\hat{a},\hat{b} = 0,1,2,3,5$. We define $\chi_a = Q_{[5a]}, a = 0,1,2,3$ and thus arrive at the sought-for $\overline{SL}(4,R)$ four-vector.

5. $\overline{SL}(3,R)$ CONTENT OF THE $\overline{SL}(4,R)$ LADDER REPRESENTATIONS

In order to study the $\overline{SL}(3,R)$ irreducible representation content of the $\overline{SL}(4,R)$ irreducible representations, it is convenient to define the following set of $\overline{SL}(4,R)$ algebra generators: The compact generators are $(p,q,r = 0,\pm 1)$

$$J_p = J_p^{(1)} + J_p^{(2)}, \quad N_p = (\frac{-p}{\sqrt{2}})^{|p|}(J_p^{(1)} - J_p^{(2)}), \quad p = 0, \pm 1,$$

while the three noncompact $\overline{SO}(3)$ irreducible tensor operators read

$$Z_p^{(2)} =< 2p|11qr > Z_{qr}, \quad Z_p^{(1)} =< 1p|11qr > Z_{qr}, \quad Z_0^{(0)} =< 00|11qr > Z_{qr}.$$

The noncompact generators of the $\overline{SL}(3,R)$ algebra are $T_p = 2Z_p^{(2)}$, while the boost generators are given by $K_p = i\sqrt{2}Z_p^{(1)}$. Moreover, in order to simplify the evaluation of the relevant matrix elements, it is convenient to introduce the operator

$$S = \sqrt{3}Z_0^{(0)},$$

that commutes with the entire $\overline{SL}(3,R)$ group.

The quantum numbers of the $\overline{SL}(4,R)$ irreducible representation decomposition w.r.t. its $\overline{SL}(3,R)$ subgroup are determined by the

$$\overline{SL}(4,R) \ \supset \ R_+ \otimes \overline{SL}(3,R) \ \supset \ R_+ \otimes \overline{SO}(3)$$

group chain. The invariant subspaces of the R_+ subgroup generator S determine the $\overline{SL}(3,R)$ subgroup invariant subspaces as well - the nontrivial question is to determine whether these $\overline{SL}(3,R)$ subspaces are irreducible or not, and finally to determine their multiplicity. As for the irreducibility question, one can make use of the $\overline{SL}(3,R)$ invariant, Casimir, operators.

We will restrict ourselves, to the case of the ladder $\overline{SL}(4,R)$ irreducible representations and consider their decomposition as given by the above subgroup chain. First of all, one can prove that invariant eigen subspaces of the R_+ generator S, characterized by fixed J, M quantum numbers of the $\overline{SO}(3)$ subgroup, are nondegenerate. One can prove this statement by showing that all vectors with the same quantum numbers span one-dimensional subspaces.

In the case of unitary irreducible representations of the $\overline{SL}(4,R)$ group, S has to be represented by a Hermitian operator in the Hilbert space and its eigenvalues are real numbers. Due to the fact that the set of J quantum number values is unlimited (in contradistinction to the finite representation case), there are no constraint on the S eigenvalues whatsoever. Indeed in each invariant subspace the eigenvalues of S, say α, are arbitrary real numbers: $S|\ \ >= \alpha|\ \ >$, $\alpha \in R$.

Owing to the fact that the $\overline{SL}(3,R)$ group generators T_p connect the ladder representation states with $\Delta J = \pm 2$, the S invariant subspaces of given α split into those of even and odd J values. The $\overline{SL}(3,R)$ Casimir operators, $C_2 = J \cdot J - \frac{1}{2} T \cdot T$ and $C_3 = J \cdot T \cdot J + \frac{1}{3} T \cdot T \cdot T$, yield the following constraints [29] on the $\overline{SL}(3,R)$ and $\overline{SL}(4,R)$ representation labels $(\alpha, \sigma_2, e_2 \in R)$

$$\sigma_1 = 0 \qquad \sigma_2 = \alpha - 3e_2$$

Finally, one finds that the ladder $\overline{SL}(4,R)$ unitary irreducible representations decompose w.r.t the $R_+ \otimes \overline{SL}(3,R)$ subgroup representations according to the following formula:

$$D^{ladd}_{\overline{SL}(4,R)}(j;0,e_2) \quad \supset$$

$$\int^{\oplus} d\alpha \{ [D_{R_+}(\alpha) \otimes D^{ladd}_{\overline{SL}(3,R)}(0;0,\alpha - 3e_2)] \oplus [D_{R_+}(\alpha) \otimes D^{ladd}_{\overline{SL}(3,R)}(1;0,\alpha - 3e_2)] \},$$

where, $j = 0, \frac{1}{2}$ and $e_2 \in R$. Thus, to conclude, the ladder unitary irreducible representations of the $\overline{SL}(4,R)$ group decompose into a direct integral of the $\overline{SL}(3,R)$ group ladder unitary irreducible representations.

6. ANHOLONOMIC AND HOLONOMIC INDICES IN GRAVITY, WORLD SPINORS

Technically, it was the unembeddability of finite $\overline{SO}(1,3)$ spinors in finite (i.e. tensor) representations of $\overline{SL}(4,R)$ that required the 1929 introduction (by Hermann Weyl and by Fock and Ivanenko) of the tetrad frames e^a for curved space-time,

$$e^a = e^a_\mu dx^\mu \ , \quad e^a_\mu(\bar{x}) \equiv \left(\partial\xi^a(x)/\partial x^\mu\right)_{x=\bar{x}} \ ,$$

with the contraction

$$e^a h_b = \delta^a_b \ , \quad h_b = h^\mu_b(x)\partial_\mu \ .$$

$\xi^a_{\bar{x}}$ is a set of coordinate axes erected at $x = \bar{x}$, locally inertial there. Gravity then involves two invariance groups: the anholonomic (tangent frame) group, here $\mathcal{L}$ and the covariance group $\overline{Diff}(4,R)$. To achieve the overall transition to a local tangent frame, we apply tetrads to the indices of a world-tensor

$$\phi^{\mu\nu\cdots}(x) \to \phi^{ab\cdots}(x) = e^a_\mu(x)e^b_\nu(x)\cdots\phi^{\mu\nu\cdots}(x) \ .$$

The tetrad indices are contracted through the Minkowski metric η_{ab}, while for world tensor indices this is achieved by the metric $g_{\mu\nu}(x)$. The two are connected via

$$\phi^a(x)\eta_{ab}\phi^b(x) = \phi^\mu(x)e^a_\mu(x)\eta_{ab}e^b_\nu(x)\phi^\nu(x) = \phi^\mu(x)g_{\mu\nu}(x)\phi^\nu(x) \ .$$

Note that the role of η_{ab} is fulfilled in the finite Dirac algebra by $\beta = \gamma^0$, for the spinor components.

The Principle of Equivalence is fulfilled for $\phi^{def\cdots}$ by the following transition from flat to curved space (Λ^c_b is a numerical matrix representation of the $\overline{SO}(1,3)$ generators on the $\phi^{def\cdots}$ basis)

$$\partial_\mu\phi \to D_a\phi = h^\mu_a(\partial_\mu - \omega^b_{\mu c}\Lambda^c_b)\phi \ ,$$

and in the opposite direction, $D_a\phi \to \partial_\mu\phi \ , \quad e^a_\mu \to \delta^a_\mu \ , \quad h^\mu_a \to \delta^\mu_a \ , \quad {\omega_\mu}^{bc} \to 0$. At the same time, for world-tensor fields $\phi^{k\lambda\nu}$

$$\partial_\mu\phi \to D_\mu\phi = [\partial_\mu - {\Gamma_\mu}^\rho{}_\sigma(\Sigma^\sigma_\rho)]\phi \ ,$$

where Σ^σ_ρ is a numerical matrix representation of the $\overline{SL}(4,R)$ generators on the $\phi^{k\lambda\nu\cdots}$ basis.

What is special about the manifields Φ and Ψ is that they do not have to be segregated in the local frame. The unirreps of $\overline{SL}(4,R)$ support $\overline{Diff}(4,R)$ and can thus be treated holonomically. Mickelsson [30] has constructed an equation for a holonomic (and non multiplicity-free) spinor in affine gravity, where the flat limit does not hold, i.e. the extinction of the gravitational field leaves a residual global $\overline{SL}(4,R)$ invariance and thereby violates the Principle of Equivalence. However, this might fit in a model in which the Lorentz group would emerge as the symmetry of flat space-time after a further (spontaneous) symmetry breakdown [24].

To consider world spinors in ordinary riemannian Einstein gravity, we denote by $\Psi^M(x), M,N = 1,2\cdots,\infty$, the M-component of the holonomic manifield, carrying a realization of $\overline{Diff}(4,R)$, the covering group of general coordinate transformations. In the local (anholonomic) frame, such a field obeys the Lorentz invariant equation, i.e. its components $\Psi^U(x), U,W = 1,2,\cdots,\infty$, correspond to the reduction of the representation $D^{\rm disc}(\frac{1}{2},0)\oplus D^{\rm disc}(0,\frac{1}{2})$ of $\overline{SL}(4,R)$ over the infinite set of representations of the compact sub-group $\overline{SO}(4)$, representing here non-unitary finite representations of $\overline{SO}(1,3)$. We now define a pseudo-frame $E^U{}_M(x)$ s.t.

$$\Psi^U(x) = E^U{}_M(x)\psi^M(x) \ .$$

The $E^U{}_M(x)$ and their inverses $H^M{}_U(x)$ are thus infinite matrices related to the quotient $\overline{Diff}(4,R)/\overline{SL}(4,R)$. Their transformation properties are

$$\delta E^U{}_M(x) = -\frac{1}{2}i\epsilon_b{}^a(x)\{\Lambda_a{}^b\}^U_V E^V{}_M(x) + \partial_\mu\xi^\rho\{\Sigma^\mu{}_\rho\}^N_M E^U{}_N(x).$$

Denoting by B the constant γ^0-like matrix in the X_μ set in the manifield wave equation we have

$$(\Psi^+(x))^U\{B\}_{UV}\Psi^V(x) \quad \rightarrow$$

$$(\Psi^+(x))^M E_M{}^U(x)\{B\}_{uv}E_N{}^V(x)\Psi^N(x) = (\Psi^+(x))^M G_{MN}(x)\Psi^N(x)$$

where $G_{MN}(x)$ is a functional of the gravitational field realizing the metric on the world-spinor components. The induced Riemannian condition, yields

$$D_\mu E^U{}_M(x) = 0 \ , \quad D_\mu G_{MN}(x) = 0 \ .$$

In the absence of other spinor fields, the above equation involves the Christoffel connection only,

$$\partial_\mu G_{MN} - \Gamma_\mu{}^\rho{}_\sigma \{\Sigma_\rho{}^\sigma\}^P_M G_{PN} - \Gamma_\mu{}^\rho{}_\sigma \{\Sigma_\rho{}^\sigma\}^P_N G_{MP} = 0 \ ,$$

which can be solved for G_{MN}, knowing Γ and Σ.

The pseudo-frame $E^U{}_M$ can be realized geometrically in an associate vector bundle over the bundle of linear frames. $E^U{}_M d\Psi^M$ is a frame on the fiber.

We now consider the infinite-dimensional representations (unirreps) of the double covering $\overline{Diff}(4,R)$ of the group of analytic diffeomorphisms. There is a rather elegant and economic method for this construction, which makes use of the pseudo-frames $E^U{}_M(x)$ and of the knowledge of the $\overline{SL}(4,R)$ unirreps.

The holonomic form of the $\overline{SO}(1,3)$ generators is given, for an arbitrary infinite-dimensional representation, by ($H = E^{-1}$)

$$(M^a{}_b)^N{}_L(x) = H^N{}_U(x)(M^a{}_b)^U{}_V E^V{}_L(x) \ .$$

In order to have a correct particle physics interpretation, we take for the $\overline{SO}(1,3)$ an infinite direct sum of finite-dimensional non-unitary representations as explained. The corresponding holonomic Lorentz-covariant matter field transforms infinitesimally as follows:

$$\delta\Psi^N(x) = i\{\xi^\mu[\delta^N_L\partial_\mu + H^N{}_U(x)\partial_\mu E^U{}_L(x)] - \frac{1}{2}\epsilon^b{}_a H^N{}_U(x)(M^a{}_b)^U{}_V E^V{}_L(x)\}\Psi^L(x) \ .$$

An $\overline{SO}(1,3)$ infinitely reducible representation, which in its turn furnishes a basis for a $\overline{SL}(4,R)$ unirrep, can now be lifted to a $\overline{Diff}(4,R)$ representation. The corresponding holonomic spinor/tensor manifields fit ordinary general relativity over a riemannian space-time. A generalization to the metric-affine theory, or even to the full general-affine theory, is rather straightforward. The anholonomic Lorentz generators are substituted by the $\overline{GL}(4,R)$ ones:

$$Q^a{}_b = \frac{1}{2}(M^a{}_b + T^a{}_b + \frac{1}{2}\delta^a_b D) \ , \quad a,b = 0,1,2,3 \ ,$$

where $T^a{}_b$ are the four-volume-preserving shear-like generators, while D generates the dilation group. The holonomic version of these generators, for an arbitrary unirrep, is

given by

$$(Q^a{}_b)^M{}_N(x) = H^M{}_U(x)(Q^a{}_b)^U{}_V E^V{}_N(x) \ .$$

The transformation properties of a holonomic spinor/tensor manifield are given as follows

$$\delta\Psi^M(x) = i\{\xi^\mu[\delta^M_N\partial_\mu + H^M{}_U(x)\partial_\mu E^U{}_N(x)] - \alpha^b_a H^M{}_U(x)(Q^a{}_b)^U{}_V E^V{}_N(x)\}\Psi^N(x) \ ,$$

where $\alpha^b{}_a$ are $\overline{SL}(4,R)$ parameters. The pseudo-frame under U runs here over a basis of an $\overline{SL}(4,R)$ unirrep. The resulting manifields transform with respect to $\overline{Diff}(4,R)$ according to the representation generated by the operators $(Q^a{}_b)^M{}_N(x)$.

An explicit construction of the $\overline{Diff}(4,R)$ unirreps requires, a knowledge of the $\overline{SL}(4,R)$ unirreps.

If we consider

$$\delta\Psi^M(x) = i\xi^\mu\{\delta^M_N\partial_\mu + H^M{}_U(x)\partial_\mu E^U{}_N(x) - i\alpha^b{}_a\partial_\mu[H^M{}_U(x)(Q^a{}_b)^U{}_V E^V{}_N(x)]\}\Psi^N(x) \ ,$$

and make an expansion of the pseudo-frames in a power series of the coordinates x^ν, we obtain the corresponding representation of the (infinite) Ogievetsky algebra, defined in the space of manifield components. This algebra is generated by $\{P_\mu, F^{\nu_1,\nu_2\cdots\nu_n}_\mu \mid n = 1,2,\cdots,\infty\}$ and the intrinsic part F of these generators is given by [31]

$$\hat{F}^{\nu_1,\nu_2\cdots\nu_n}_\mu = \partial_\rho(x^{\nu_1}x^{\nu_2}\cdots x^{\nu_n})h^\rho{}_a e^b{}_\mu(Q^a{}_b)^U{}_V \ .$$

Substituting here the generator matrix elements $(Q^a{}_b)^U{}_V$ of an $\overline{\mathrm{SL}}$(4,R) unirrep we obtain the matrix elements of the Ogievetsky algebra for the corresponding algebraic representation of the $\overline{Diff}(4,R)$ group.

We close this review with a remark about possible future new applications of world spinors. Should the Quantum Superstring indeed "take over" as the fundamental theory of (Quantum) Gravity, it seems that the geometry beyond the Planck energy might well be nonriemannian. The structure of string theory already involves infinite linear representations, those of $Diff(R^2)$. With the recent explosion in "dual" systems, in which superstrings become just one special case of "extendons" ("p-branes") of dimensionalities $p \leq 6$ (in $D = 11$, for instance), affine (or metric-affine) constructions might become the most convenient tool in dealing with systems supporting the action of $Diff(R^p)$ [32].

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